J. Fluid Mech. (1991), *vol.* 223, *pp.* 357–362
Printed in Great Britain

The exact solution to Phillips' equation for the degree of saturation of short waves in the presence of ocean currents

By S. J. HOGAN
University of Oxford, Mathematical Institute, 24–29 St Giles, Oxford OX1 3LB, UK

(Received 2 October 1989 and in revised form 27 June 1990)

An equation, derived by Phillips (1984) to describe the variation of a wind-wave spectrum in the presence of a current, is shown to have an exact solution. For certain choices of current, the results are shown to simplify even further.

1. Introduction

Bottom topography and internal waves induce current variations at the ocean surface. These in turn modulate any surface wave field already present, created for example by the wind. When an active radar probes the surface, the radiation is returned by several types of scattering mechanism, dependent on the angle of the incidence ϕ, measured from the vertical. For ϕ away from grazing and away from normal incidence, Bragg reflection gives a return signal from free waves of wavenumber k satisfying the condition

$$k = 2k_r \sin\phi, \tag{1.1}$$

where k_r is the wavenumber of the incident electromagnetic radiation. Thus for L-band radar with a frequency around 1 GHz and ϕ around 60°, the surface waves responsible for the backscatter are in the decameter range (0.1–1 m). The effect of currents on these short waves is the subject of this paper. Recent work in this field can be found in either Komen & Oost (1989) or the full reports of the Synthetic Aperture Radar Internal Wave Signature Experiment (SARSEX) (1988).

Phillips (1984) has derived an equation which describes the patterns of spectral variation due to the presence of a current. In this paper, we show that the equation has an exact solution which can be further simplified for certain analytic choices of current distribution. The equation and its derivation are outlined in §2. The solution procedure is given in §3 and two different choices of current are considered in §4.

2. Phillips' equation

The wave dynamics is described by an equation for the action spectral density $N(\boldsymbol{k})$, where $\boldsymbol{k}$ is the wavenumber, which is related to the surface displacement spectrum $\Psi(\boldsymbol{k})$ by

$$N(\boldsymbol{k}) = \frac{g}{\sigma}\Psi(\boldsymbol{k}) = \left(\frac{g}{k}\right)^{\frac{1}{2}}\Psi(\boldsymbol{k}), \tag{2.1}$$

where $\sigma = (gk)^{\frac{1}{2}}$ is the intrinsic frequency of free gravity waves.

The equation for $N(\boldsymbol{k})$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}N(\boldsymbol{k}) = \frac{\partial N}{\partial t}+(\boldsymbol{c}_{\mathrm{g}}+\boldsymbol{U})\cdot\boldsymbol{\nabla}N = \Sigma \tag{2.2}$$

along rays, where $\boldsymbol{c}_{\mathrm{g}} = \partial\sigma/\partial\boldsymbol{k}$ is the group velocity, $\boldsymbol{U}$ is the surface current velocity and Σ is a sum of source terms attributed to such mechanisms as wind input, nonlinear wave–wave interactions and dissipation by wave breaking or generation of parasitic capillaries.

We also have the equation of conservation of wave crests, or kinematical conservation equation,

$$\frac{\partial \boldsymbol{k}}{\partial t}+\boldsymbol{\nabla}(\sigma+\boldsymbol{k}\cdot\boldsymbol{U}) = 0 \tag{2.3}$$

together with the condition

$$\boldsymbol{\nabla}\wedge\boldsymbol{k} = 0. \tag{2.4}$$

Phillips (1984) argues that, for short waves in the range 0.1–1 m, an equilibrium (that is $\mathrm{d}N/\mathrm{d}t = 0$) can be achieved in a region devoid of currents by a balance between wind input and dissipation, with nonlinear wave–wave interactions being negligible. He uses Plant's (1982) expression for the wind input and argues for a functional dependence of the dissipation on the local spectral density.

The presence of a current induced either by bottom topography or by internal waves then perturbs that equilibrium. Phillips (1984) derives an equation which describes these changes. It turns out that a better representation is not in terms of the action $N(\boldsymbol{k})$ but the non-dimensional degree of saturation B defined by

$$B = g^{-\frac{1}{2}}k^{\frac{9}{2}}N(\boldsymbol{k}) = k^4\Psi(\boldsymbol{k}). \tag{2.5}$$

The equation is obtained from our equations (2.1)–(2.5) together with the relevant form for the source terms Σ. It is given by (Phillips 1984, equation (4.1))

$$\frac{\partial B}{\partial t}+(c_{\mathrm{g}j}+U_j)\frac{\partial B}{\partial x_j}+\frac{9}{2}\frac{k_i k_j}{k^2}\frac{\partial U_i}{\partial x_j}B-k_j\frac{\partial U_i}{\partial x_j}\frac{\partial B}{\partial k_i} = \sigma m\left(\frac{u_*}{c}\right)^2\left\{1-\left(\frac{B}{B_0}\right)^{n-1}\right\}B, \tag{2.6}$$

where $m = 0.04\cos\theta$, θ the angle between $\boldsymbol{k}$ and the wind, u_* the friction velocity, $c = (g/k)^{\frac{1}{2}}$ the phase speed of the short waves and n a number between 3 and 5. B_0 is the value of B at equilibrium. The first term on the right represents wind input and the second represents dissipation. The wavenumber $\boldsymbol{k}$ is regarded as fixed because of the Bragg scattering condition (1.1).

For current variations of magnitude U_0 which occur over a lengthscale L and with

$$\boldsymbol{U} = U_0\boldsymbol{f}(\boldsymbol{x}/L) = U_0\boldsymbol{f}(\boldsymbol{\xi}) \tag{2.7}$$

then the steady form of (2.6) can be conveniently written in terms of $b(\boldsymbol{\xi}) = B/B_0$, the local relative degree of saturation, as

$$\left[\frac{c_{\mathrm{g}j}}{c}+\frac{U_0}{c}f_j(\boldsymbol{\xi})\right]\frac{\partial b}{\partial \xi_j}+\frac{9}{2}\frac{U_0}{c}\frac{k_i k_j}{k^2}\frac{\partial f_i}{\partial \xi_j}b-\frac{U_0}{c}k_j\frac{\partial f_i}{\partial \xi_j}\frac{\partial b}{\partial k_i} = 2\pi m S\{1-b^{n-1}\}b. \tag{2.8}$$

The sensing parameter S is defined by

$$S = \frac{L}{\lambda}\left(\frac{u_*}{c}\right)^2 = \frac{1}{2m}\left(\frac{L}{c}\right)\left\{m\sigma\left(\frac{u_*}{c}\right)^2\right\}. \tag{2.9}$$

It represents the ratio of the time taken for a wave packet to cross the current variation to the growth time of waves by wind. If S is large, b is small and vice versa.

If we take $\boldsymbol{f} = (f, 0, 0)$ and $\boldsymbol{\xi} = (\xi, 0, 0)$ to be in the same direction and assume no dependence of b on $\boldsymbol{k}$, the final equation then becomes

$$\left(\frac{1}{2}+\frac{U_0}{c}f(\xi)\right)\frac{\mathrm{d}b}{\mathrm{d}\xi}+\frac{9}{2}\frac{u_0}{c}\frac{\mathrm{d}f}{\mathrm{d}\xi}b = 2\pi m\,S(1-b^{n-1})\,b. \tag{2.10}$$

Phillips (1984) chose a particular form for $f(\xi)$ and integrated (2.10) numerically. For $S = 0$, he showed that (2.10) has the exact solution

$$b(\xi) = \left[1+\frac{2U_0 f(\xi)}{c}\right]^{-\frac{9}{2}} \tag{2.11}$$

subject to the condition that $b = 1$ when there is no current (i.e. $f = 0$).

We shall show in the next section that (2.10) has an exact solution for $S \neq 0$.

3. Exact solution

Let us rewrite (2.10) in the form

$$\frac{\mathrm{d}b}{\mathrm{d}\xi} = A_1(\xi)\,b - A_n(\xi)\,b^n, \tag{3.1}$$

where
$$A_1(\xi) = A_n(\xi) - \frac{9U_0 f'(\xi)}{c+2U_0 f(\xi)} \tag{3.2}$$

and
$$A_n(\xi) = \frac{4\pi mcS}{c+2U_0 f(\xi)}. \tag{3.3}$$

Equation (3.1) is called Bernoulli's equation (Davis 1962, p. 49). It is a natural generalization of Riccati's equation, for which $n = 2$. In fact for $n = 3$, the case integrated by Phillips, it is called Abel's equation (see for example Davis 1962, chapter 3, section 9). It is solved exactly as follows. Substitute

$$b(\xi) = \frac{1}{p(\xi)\,z(\xi)} \tag{3.4}$$

into (3.1), to give
$$-\frac{1}{pz^2}\frac{\mathrm{d}z}{\mathrm{d}\xi} - \frac{1}{zp^2}\frac{\mathrm{d}p}{\mathrm{d}\xi} = \frac{A_1}{pz} - \frac{A_n}{p^n z^n}. \tag{3.5}$$

Let us now choose, once and for all

$$\frac{\mathrm{d}p}{\mathrm{d}\xi} = -A_1(\xi)\,p(\xi) \tag{3.6}$$

and hence
$$p(\xi) = p_c \exp\left[-\int_{\xi_c}^{\xi} A_1(\eta)\,\mathrm{d}\eta\right], \tag{3.7}$$

where $p(\xi_c) = p_c$. Equation (3.5) then simplifies to become

$$z^{n-2}\frac{\mathrm{d}z}{\mathrm{d}\xi} = \frac{A_n}{p^{n-1}} \tag{3.8}$$

and hence
$$[z(\xi)]^{n-1} = (n-1)\int_{\xi_c}^{\xi} \frac{A_n(\zeta)}{[p(\zeta)]^{n-1}}\,\mathrm{d}\zeta + z_c^{n-1}, \tag{3.9}$$

where $z(\xi_c) = z_c$.

We now combine (3.4), (3.7) and (3.9) to give the exact solution of (3.1):

$$b(\xi) = \frac{b_c \exp\left[\int_{\xi_c}^{\xi} A_1(\eta)\,\mathrm{d}\eta\right]}{\left[1+(n-1)\,b_c^{n-1}\int_{\xi_c}^{\xi} A_n(\zeta)\exp\left[(n-1)\int_{\xi_c}^{\zeta} A_1(\eta)\,\mathrm{d}\eta\right]\mathrm{d}\zeta\right]^{1/n-1}} \tag{3.10}$$

where $b(\xi_c) = b_c$.

The form of this equation is similar to that found by Hughes (1978). He derived a widely used equation to describe perturbations to the action $N(\boldsymbol{k})$ in the presence of a current. He took $n = 2$, that is Riccati's equation. His notion of quiescence, viz. $1/N(\boldsymbol{k})$, coincides with the form of the substitution needed to solve the case $n = 2$. For the case of arbitrary n, a more general type of substitution, our (3.4), is needed.

Clearly when $S = 0$, $A_n(\xi) \equiv 0$ and hence

$$b(\xi) = b_c \exp\left[-\int_{\xi_c}^{\xi} \frac{9U_0 f'(\eta)}{c+2U_0 f(\eta)}\,\mathrm{d}\eta\right]. \tag{3.11}$$

The integration can be done exactly by noting that $f'(\eta)\,\mathrm{d}\eta = \mathrm{d}f$ and hence, since $b = 1$ when $f = 0$, we recover (2.11). We conclude this section by noting that (3.10) can also be obtained from (3.1) by the substitution

$$b(\xi) = [q(\xi)]^{\beta} \tag{3.12}$$

and taking $\beta = 1/(1-n)$.

4. Different forms of current function $f(\xi)$

It is possible to go further and, for certain analytic choices of current function $f(\xi)$, produce an exact form for part of the expression for $b(\xi)$, our (3.10).

In general, from (3.2) and (3.3),

$$\int^{\xi} A_1(\eta)\,\mathrm{d}\eta = 4\pi mcS\int^{\xi}\frac{\mathrm{d}\eta}{c+2U_0 f(\eta)} - \tfrac{9}{2}\log_e|c+2U_0 f(\xi)|. \tag{4.1}$$

This expression provides us with an important element of (3.10). Phillips (1984) chose a current of the form

$$f(\xi) = \tfrac{1}{2}(1+\tanh\xi). \tag{4.2}$$

In this case the integral on the right-hand side of (4.1) can be found explicitly from tables (Gradshteyn & Ryzhik 1965) for $c > 0$ to be

$$\int^{\xi}\frac{\mathrm{d}\eta}{c+U_0+U_0\tanh\eta} = \frac{(c+U_0)\,\xi - U_0\log_e\cosh[\xi+\tanh^{-1}(U_0/(c+U_0))]}{c(c+2U_0)} \quad (U_0 > -\tfrac{1}{2}c). \tag{4.3}$$

$$= \frac{(c+U_0)\,\xi - U_0\log_e\sinh[\xi+\tanh^{-1}(c+U_0/U_0)]}{c(c+2U_0)} \quad (U_0 < -\tfrac{1}{2}c) \tag{4.4}$$

and hence

$$H(\xi, K) = \exp\left[K \int^{\xi} A_1(\eta)\,\mathrm{d}\eta\right] = \exp\left[K \frac{4\pi m S(c+U_0)}{c+2U_0}\xi\right]$$

$$\times \left\{\cosh\left[\xi + \tanh^{-1}\left(\frac{U_0}{c+U_0}\right)\right]\right\}^{-4\pi m S U_0 K/(c+2U_0)} [c+U_0+U_0\tanh\xi]^{-\frac{9}{2}K} \quad (U_0 > -\tfrac{1}{2}c) \tag{4.5}$$

$$= \exp\left[K \frac{4\pi m S(c+U_0)}{c+2U_0}\xi\right]$$

$$\times \left\{\sinh\left[\xi + \tanh^{-1}\left(\frac{c+U_0}{U_0}\right)\right]\right\}^{-4\pi m S U_0 K/(c+2U_0)} [c+U_0+U_0\tanh\xi]^{-\frac{9}{2}K} \quad (U_0 < -\tfrac{1}{2}c), \tag{4.6}$$

where $K = 1$ for the numerator, $K = n-1$ for the denominator. Therefore

$$b(\xi) = \frac{b_c[H(\xi,1)/H(\xi_c,1)]}{\left[1 + 4\pi m c(n-1)\, S b_c^{n-1} \int_{\xi_c}^{\xi} \frac{[H(\zeta, n-1)/H(\xi_c, n-1)]}{[c+U_0+U_0\tanh\zeta]}\mathrm{d}\zeta\right]^{1/n-1}}. \tag{4.7}$$

Equations (4.3)–(4.7) provide the complete solution for $b(\xi)$ when $f(\xi)$ is given by (4.2). Only one numerical integration is needed.

Another choice of $f(\xi)$ would be

$$f(\xi) = \tfrac{1}{2}(1+\sin 2\pi\xi). \tag{4.8}$$

In this case, for $c > 0$,

$$\int^{\xi} \frac{\mathrm{d}\eta}{c+U_0+U_0\sin 2\pi\eta} = \frac{1}{\pi[c(c+2U_0)]^{\frac{1}{2}}}\tan^{-1}\left[\frac{(c+U_0)\tan^{-1}\pi\xi + U_0}{[c(c+2U_0)]^{\frac{1}{2}}}\right] \quad (U_0 > -\tfrac{1}{2}c) \tag{4.9}$$

$$= \frac{1}{2\pi[-c(c+2U_0)]^{\frac{1}{2}}}\log_e\left[\frac{(c+U_0)\tan^{-1}\pi\xi + U_0 - [-c(c+2U_0)]^{\frac{1}{2}}}{(c+U_0)\tan^{-1}\pi\xi + U_0 + [-c(c+2U_0)]^{\frac{1}{2}}}\right] \quad (U_0 < -\tfrac{1}{2}c) \tag{4.10}$$

and hence

$$H(\xi, K) = \exp\left[K \int^{\xi} A_1(\eta)\,\mathrm{d}\eta\right]$$

$$= \exp\left\{4mSK\left(\frac{c}{c+2U_0}\right)^{\frac{1}{2}}\tan^{-1}\left[\frac{(c+U_0)\tan^{-1}\pi\xi + U_0}{[c(c+2U_0)]^{\frac{1}{2}}}\right]\right\} \times [c+U_0+U_0\sin 2\pi\xi]^{-9K/2}. \quad (U_0 > -\tfrac{1}{2}c) \tag{4.11}$$

$$= \left[\frac{(c+U_0)\tan^{-1}\pi\xi + U_0 - [-c(c+2U_0)]^{\frac{1}{2}}}{(c+U_0)\tan^{-1}\pi\xi + U_0 + [-c(c+2U_0)]^{\frac{1}{2}}}\right]^{2mSK(-c/(c+2U_0))^{\frac{1}{2}}} [c+U_0+U_0\sin 2\pi\xi]^{-9K/2} \quad (U_0 < -\tfrac{1}{2}c) \tag{4.12}$$

Therefore

$$b(\xi) = \frac{b_c[H(\xi,1)/H(\xi_c,1)]}{\left[1 + 4\pi m c(n-1)\, S b_c^{n-1} \int_{\xi_c}^{\xi} \frac{[H(\zeta, n-1)/H(\xi_c, n-1)]}{[c+U_0+U_0\sin 2\pi\zeta]}\mathrm{d}\zeta\right\}^{1/n-1}}. \tag{4.13}$$

Equations (4.9)–(4.13) are the complete solution for $b(\xi)$ when $f(\xi)$ is given by (4.8). Care must be exercised when evaluating $b(\xi)$ for $f(\xi)$ negative.

5. Summary

We have shown that Phillips (1984) equation (2.10) for the local relative degree of saturation $b(\xi)$ has an exact solution given by equation (3.10). For certain choices of current function $f(\xi)$, the formula for $b(\xi)$ can be simplified even further to produce an expression which requires only one numerical integration (e.g. (4.7) or (4.13)).

The author carried out this work whilst holding the CEGB Research Fellowship in Applied Mathematics at St Catherine's College, Oxford.

REFERENCES

DAVIS, H. T. 1962 *Introduction to Nonlinear Differential and Integral Equations*. Dover.

GRADSHTEYN, I. S. & RYZHIK, I. M. 1965 *Tables of Integrals, Series and Products*, 4th edn. Academic.

HUGHES, B. A. 1978 The effect of internal waves on surface wind waves. 2. Theoretical analysis. *J. Geophys. Res.* **83**, 455–465.

KOMEN, G. J. & OOST, W. A. 1989 (Eds.) *Radar Scattering from Modulated Wind Waves*. Kluwer.

PHILLIPS, O. M. 1984 On the response of short ocean wave components at a fixed wavenumber to ocean current variations. *J. Phys. Oceanogr.* **14**, 1425–1433.

PLANT, W. J. 1982 A relationship between wind stress and wave slope. *J. Geophys. Res.* **87**, 1961–1967.

SAR Internal Wave Signature Experiments (SARSEX) 1988 *J. Geophys. Res.* **93**, 12217–12380.